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GENERALIZED EFFECTIVE STIFFNESS THEORY FOR THE MODELING OF FIBER-REINFORCED COMPOSITES

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Abstract—Effective stiffness theory of the Nth order is derived for the modeling of the three-dimensional time-dependent motion of a fiber-reinforced composite. The fibers are assumed to be of a rectangular cross section and are imbedded in the matrix in the form of a doubly periodic array. The resulting theory represents the composite as a higher order homogeneous continuum with microstructure whose motion is governed by higher order displacements. The derivation is systematic and can be applied to elastic as well as anelastic composites to the desired degree of accuracy.

INTRODUCTION

The effective stiffness theory was developed to model laminated composites and fiber-reinforced composites as higher order continua with microstructure, see the monograph by Achenbach[1] and references cited there. A great advantage of this theory is its ability to permit the application of a general type of loading to a three-dimensional configuration and no symmetry conditions need to be assumed, thus allowing an arbitrary three-dimensional motion of the composite.

In [1] several first order effective stiffness theories are discussed, and a second order effective stiffness theory for elastic laminated materials is reviewed in [2]. A generalization to Nth order theory for the modeling of anelastic and elastic-viscoplastic laminated composites appears in [3] where the governing equations are derived directly from the basic field equations without invoking a variational principle.

Effective stiffness theory for fiber-reinforced composites is much more difficult to derive due to the different geometries involved in their structure. A representative cell in the material (assuming a periodic arrangement of the fibers within the matrix) contains a circle representing the fiber cross section and a rectangle or a hexagon for the surrounding matrix. Accordingly, a systematic development of a higher order effective stiffness theory is complicated and in a second order theory it is necessary to assume in advance different displacement distributions in different circumstances, see [1]. Hlavacek presented in [4] a second order theory, but it involves several simplifying assumptions. For example, the continuity of the stresses across the interface between the phases is not guaranteed, and that a linear expression for the displacement distribution within the fibrous material would be sufficient. In both Refs. [1, 4] the equations of motion are derived from a Lagrangian function upon which the Hamilton's variational principle is applied, so that generalizations to anelastic or inelastic composites would be very difficult.

It is possible, however, to represent the circular fibers in the composite by fibers of rectangular cross sections so that a representative cell contains this time the same geometry for the fiber and the surrounding matrix. The replacement is possible since the dispersion curves for bars of circular and square cross sections turn out to be nearly the same if their cross sectional areas are approximately the same [5]. Motivated by this observation, Bartholomew and Torvik[6] developed a first order effective stiffness theory for filamentary composite materials. Their derivation is based on the construction of strain and kinetic energy densities and subsequent use of Hamilton's principle.

In this paper a generalized effective stiffness theory of the N-th order is proposed for the modeling of a composite material made of a matrix reinforced by a doubly periodic array of rectangular fibers. The equations of motions in this theory are derived directly from the basic

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field equations of the constituents so that the theory is essentially applicable to elastic as well as anelastic fiber-reinforced composites. The formulation of the boundary conditions for applied tractions is given in the framework of the derived theory. The derivation of the higher order continuum theory is systematic and can be carried out to the desired degree of accuracy.

The theory is illustrated by computing the acoustical dispersion curve for longitudinal harmonic waves propagating in the fibers direction. It is found that excellent agreement exists between the theoretical and numerical results, based on a finite element solution for a matrix reinforced by circular fibers, reported in [7] as well as experimental results. This agreement shows that the modeling of circular fibers by rectangular ones is satisfactory. In the limiting case of vanishingly small wave number the effective wave speed in the composite, whose value is known from the effective modulus theory, is obtained. Accordingly, the relevant effective moduli can be computed directly from the implicit expression provided by the frequency equation in this limiting special case.

Results are also given for an impacted fiber-reinforced slab on which a uniform time-dependent load is applied in the fibers direction.

GEOMETRY AND DISPLACEMENT EXPANSIONS

Consider a composite material consisting of a matrix constituent reinforced by unidirectional fibers of rectangular cross section. It is assumed that the fibers extends along the x_1-direction and arranged in a doubly periodic array in the x_2 and x_3 directions, see Fig. 1. In this paper we consider matrix and fibers made of perfectly elastic materials although extensions to anelastic constituents can be made.

Let d_1, h_1 denote the dimensions of the rectangular cross section of the fibers and d_2, h_2 represent the spacing of the fibers within the matrix in the x_2 and x_3 directions respectively. Due to the assumed periodic arrangement of the fibers, we consider a representative cell shown in Fig. 2. The cell is divided into four subcells $\alpha, \beta = 1, 2$, and four local systems of coordinates $x_1, \bar{x}_2^{(\alpha)}, \bar{x}_3^{(\beta)}$ are introduced whose origins are located in the center of each subcell. Their positions are denoted by $x_2^{(\alpha)}, x_3^{(\beta)}$.†

The essential step of the effective stiffness theory (EST) is the expansion of the displacement vector in each subcell in terms of the distances from the center. This expansion can be expressed in terms of the Legendre polynomials permitting the modeling of increasing complex deformation patterns within the cell. For an Nth order theory the displacement components at

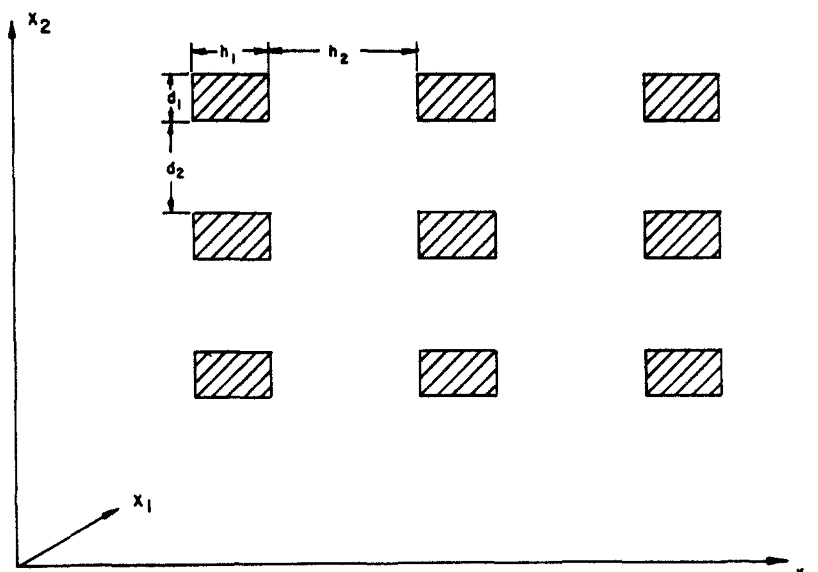

Fig. 1. A fiber-reinforced composite in which the fibers are arranged in a doubly periodic array.

†Here and in the sequel the subscripts or superscripts α, β will indicate that quantities belong to one of the subcells. Repeated α or β do not imply summation.

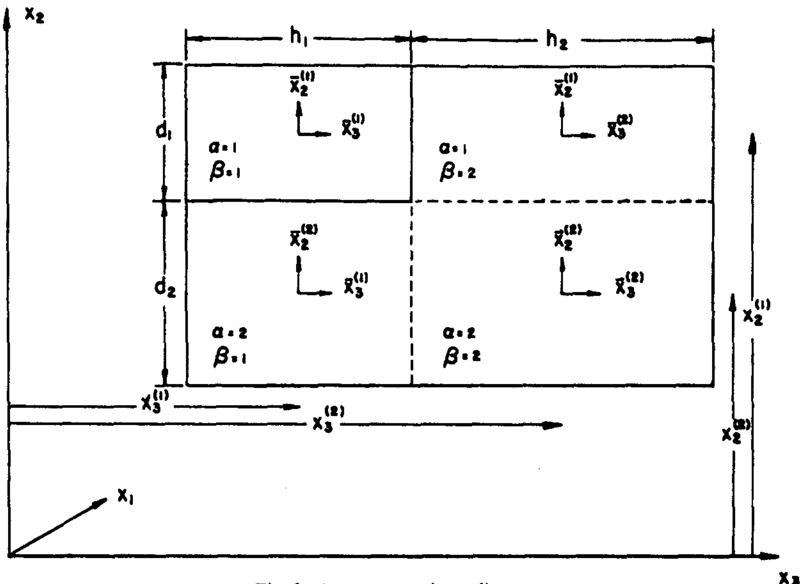

Fig. 2. A representative cell.

any point within the subcell can be expressed as

$$u_i^{(\alpha\beta)} = U_{i(m,n)}^{(\alpha\beta)}(x_1, x_2^{(\alpha)}, x_3^{(\beta)}, t)P_m(z_2^{(\alpha)})P_n(z_3^{(\beta)}) \qquad i = 1,2,3 \tag{1}$$

where $z_2^{(\alpha)} = \bar{x}_2^{(\alpha)}/(d_\alpha/2)$, $z_3^{(\beta)} = \bar{x}_3^{(\beta)}/(h_\beta/2)$, t is the time, P_m is the Legendre polynomial of order m and summation is implied by the repeated indices $m, n = 0, 1, \ldots, N$.

The displacement distribution in the composite material is described by the higher order displacements $U_{i(m,n)}^{(\alpha\beta)}$ and these are defined only on discrete lines $x_2 = x_2^{(\alpha)}$ and $x_3 = x_3^{(\beta)}$. The derivation of the continuum theory requires a smoothing operation by which the discrete nature of the composite is eliminated. This transition to the homogeneous model is performed by considering $U_{i(m,n)}^{(\alpha\beta)}$ as continuous functions of x_2 and x_3, and whose values at $x_2 = x_2^{(\alpha)}$ and $x_3 = x_3^{(\beta)}$ coincide with the values of the actual values at the centerlines of the subcell. This transition to the continuous field is indicated by writing $U_{i(m,n)}^{(\alpha\beta)}(x_1, x_2, x_3, t)$ instead of $U_{i(m,n)}^{(\alpha\beta)}(x_1, x_2^{(\alpha)}, x_3^{(\beta)}, t)$, with similar replacements for the other field variables. Consequently, both types of materials are assumed to exist simultaneously at every point of the continuum model.

The components of the small strain tensor are given by

$$\epsilon_{ij}^{(\alpha\beta)} = \tfrac{1}{2}[\partial_i u_j^{(\alpha\beta)} + \partial_j u_i^{(\alpha\beta)}] \qquad i, j = 1,2,3 \tag{2}$$

where differentiation in the x_2 and x_3 directions should be with respect to the local coordinates $\bar{x}_2^{(\alpha)}$ and $\bar{x}_3^{(\beta)}$ respectively.

INTERFACE CONTINUITY CONITIONS

At the interfaces of the subcells the displacements and normal and shear stresses must be continuous, i.e.

$$u_i^{(1\beta)}\Big|_{\bar{x}_2^{(1)} = \mp d_1/2} = u_i^{(2\beta)}\Big|_{\bar{x}_2^{(2)} = \pm d_2/2} \tag{3}$$

$$u_i^{(\alpha 1)}\Big|_{\bar{x}_3^{(1)} = \pm h_1/2} = u_i^{(\alpha 2)}\Big|_{\bar{x}_3^{(2)} = \mp h_2/2} \tag{4}$$

$$\sigma_{2i}^{(1\beta)}\Big|_{\bar{x}_2^{(1)} = \mp d_1/2} = \sigma_{2i}^{(2\beta)}\Big|_{\bar{x}_2^{(2)} = \pm d_2/2} \tag{5}$$

$$\sigma_{3i}^{(\alpha 1)}\Big|_{\bar{x}_3^{(1)} = \pm h_1/2} = \sigma_{3i}^{(\alpha 2)}\Big|_{\bar{x}_3^{(2)} = \mp h_2/2} \tag{6}$$

where $\sigma_{ij}^{(\alpha\beta)}$ are the stress components in the subcell, and the plus-and-minus signs in (3) and (5) denote the two different equations obtained depending upon whether the interface follows the subcells (1β) or (2β). Similarly the plus-and-minus signs in (4) and (6) denote the two different situations whether the interface follows $(\alpha 1)$ or $(\alpha 2)$.

The resulting eqns (3)–(6) need to insure the continuity of displacements and stresses at the interfaces between the subcells as well as between the neighboring cells. In the exact solution of harmonic wave propagation problem in a periodically layered body, Delph *et al.*[11] showed that the dependent variables are, in general, not periodic but have a quasi-periodic property. This follows from Bloch's theorem and the periodicity of the medium. Consequently, although the continuity of displacements and stresses can be satisfied across the layers interface of the subcell, the continuity across the interface between neighboring cells can be insured by using the quasi-periodic relationships. Utilizing these relations, the continuity conditions in the problem treated in [11] can be imposed in terms of displacement and stresses of the unit cell. In the present problem of a doubly periodic structure, eqns (3)–(6) insure the continuity of displacements and stresses across the subcell interfaces, but do not, in general, suffice for continuity between neighboring cells. This lack of continuity must be regarded as a part of the approximation involved in the model.

Using (1), the displacement continuity conditions (3)–(4) can be rewritten as

$$U_{i(m,n)}^{(1\beta)}(x_1, x_2^{(1)} \mp d_1/2, x_3^{(\beta)}, t)P_m(\mp 1)P_n(z_3^{(\beta)}) = U_{i(m,n)}^{(2\beta)}(x_1, x_2^{(2)} \pm d_2/2, x_3^{(\beta)}, t)P_m(\pm 1)P_n(z_3^{(\beta)}), \quad (7)$$

$$U_{i(m,n)}^{(\alpha 1)}(x_1, x_2^{(\alpha)}, x_3^{(1)} \pm h_1/2, t)P_m(z_2^{(\alpha)})P_n(\pm 1) = U_{i(m,n)}^{(\alpha 2)}(x_1, x_2^{(\alpha)}, x_3^{(2)} \mp h_2/2, t)P_m(z_2^{(\alpha)})P_n(\mp 1). \quad (8)$$

Rather than requiring (7)–(8) to be satisfied at every point along the interfaces, we impose the condition that the average displacement is continuous at the interfaces. This is achieved by integrating both sides of (7) and (8) with respect to $\bar{x}_3^{(\beta)}$ along $-1 \le z_3^{(\beta)} \le 1$ and $\bar{x}_2^{(\alpha)}$ along $-1 \le z_2^{(\alpha)} \le 1$ respectively. This gives

$$U_{i(m,0)}^{(1\beta)}(x_1, x_2^{(1)} \mp d_1/2, x_3^{(\beta)}, t)P_m(\mp 1) = U_{i(m,0)}^{(2\beta)}(x_1, x_2^{(2)} \pm d_2/2, x_3^{(\beta)}, t)P_m(\pm 1), \quad (9)$$

$$U_{i(0,n)}^{(\alpha 1)}(x_1, x_2^{(\alpha)}, x_3^{(1)} \pm h_1/2, t)P_n(\pm 1) = U_{i(0,n)}^{(\alpha 2)}(x_1, x_2^{(\alpha)}, x_3^{(2)} \mp h_2/2, t)P_n(\mp 1). \quad (10)$$

In the framework of the higher order homogeneous model it will be necessary to apply (9)–(10) simultaneously throughout the medium, since it is assumed that both types of materials and interfaces exist at every point. By performing this transition, the continuum continuity equations which correspond to (9)–(10) are obtained by expanding the resulting continuous functions about x_1, x_2, x_3, t in terms of $d_\alpha/2$ and $h_\alpha/2$ up to the Nth power:

$$\frac{1}{l!}(\mp d_1/2)^l \partial_2^l U_{i(m,0)}^{(1\beta)} P_m(\mp 1) = \frac{1}{l!}(\pm d_2/2)^l \partial_2^l U_{i(m,0)}^{(2\beta)} P_m(\pm 1) \quad (11)$$

$$\frac{1}{l!}(\pm h_1/2)^l \partial_3^l U_{i(0,n)}^{(\alpha 1)} P_n(\pm 1) = \frac{1}{l!}(\mp h_2/2)^l \partial_3^l U_{i(0,n)}^{(\alpha 2)} P_n(\mp 1). \quad (12)$$

In (11)–(12) the summations on l, m, n are from 0 to N, and $U_{i(m,n)}^{(\alpha\beta)} = U_{i(m,n)}^{(\alpha\beta)}(x_1, x_2, x_3, t)$.

The transition from the discrete to the homogeneous model replaces the stress continuity of stresses (5)–(6) by

$$\sigma_{2i}^{(1\beta)}(x_k, \bar{x}_2^{(1)} = \mp d_1/2, \bar{x}_3^{(\beta)}, t) = \sigma_{2i}^{(2\beta)}(x_k, \bar{x}_2^{(2)} = \pm d_2/2, \bar{x}_3^{(\beta)}, t), \quad (13)$$

$$\sigma_{3i}^{(\alpha 1)}(x_k, \bar{x}_2^{(\alpha)}, \bar{x}_3^{(1)} = \pm h_1/2, t) = \sigma_{3i}^{(\alpha 2)}(x_k, \bar{x}_2^{(\alpha)}, \bar{x}_3^{(2)} = \mp h_2/2, t). \quad (14)$$

Equations (11)–(12) are 24 conditions for the continuity of the displacements and (13)–(14) are 24 conditions for the continuity of the stresses.

EQUATIONS OF MOTION

The equations of motion for the homogeneous continuum which models the composite material are derived directly from the corresponding dynamic equations in the subcell regions. This derivation does not rely, therefore, on the construction of a Lagrangian function (where the continuity relations are included through the use of Lagrangian multipliers) on which Hamilton's principle is applied. Accordingly, it is possible to apply the proposed approach to anelastic fiber reinforced materials without invoking any variational principle.

The stress equations of motion in the subcell $(\alpha\beta)$, in the absence of body forces, are given by

$$\partial_i \sigma_{ij}^{(\alpha\beta)} = \rho_{\alpha\beta} \ddot{u}_j^{(\alpha\beta)} \quad |\bar{x}_2^{(\alpha)}| \le d_\alpha/2, |\bar{x}_3^{(\beta)}| \le h_\beta/2 \tag{15}$$

where $\rho_{\alpha\beta}$ is the mass density of the material in the subcell, dots represents differentiation with respect to time and differentiation in the 2 and 3 directions should be with respect to $(\bar{x}_2^{(\alpha)})$ and $(\bar{x}_3^{(\beta)})$ respectively.

In a periodically layered medium the stress resultants and the moments of the stresses over the thicknesses were used in [1] in conjunction with the balance equations of linear momentum to derive a first order EST. This was extended in [3] for the derivation of a generalized EST. In the present problem of a fiber-reinforced composite the N-th order EST the equations of motion are obtained by multiplying (15) by $(\bar{x}_2^{(\alpha)})^p(\bar{x}_3^{(\beta)})^q$, $(p, q = 0,1,\ldots, N)$, and integrating both sides with respect to $\bar{x}_2^{(\alpha)}$ and $\bar{x}_3^{(\beta)}$. This yield after integrations by parts and using (1) the following set of equations

$$\begin{aligned}\partial_1 S_{1j(p,q)}^{(\alpha\beta)} + I_{2j(p,q)}^{(\alpha\beta)} + J_{3j(p,q)}^{(\alpha\beta)} - pS_{2j(p-1,q)}^{(\alpha\beta)} - qS_{3j(p,q-1)}^{(\alpha\beta)} \\ = \rho_{\alpha\beta}(d_\alpha/2)^p(h_\beta/2)^q \ddot{U}_{j(m,n)}^{(\alpha\beta)}\delta_m^{(p)}\delta_n^{(q)}/[(2m+1)(2n+1)]\end{aligned}$$

$$j = 1,2,3 \qquad m, n, p, q = 0, 1, \ldots, N \qquad p, q \text{ are not summed.} \tag{16}$$

In (16):

$$S_{ij(p,q)}^{(\alpha\beta)} = \frac{1}{A_{\alpha\beta}} \int_{-d_\alpha/2}^{d_\alpha/2} \int_{-h_\beta/2}^{h_\beta/2} \sigma_{ij}^{(\alpha\beta)} (\bar{x}_2^{(\alpha)})^p (\bar{x}_3^{(\beta)})^q \, d\bar{x}_2^{(\alpha)} \, d\bar{x}_3^{(\beta)}, \tag{17}$$

$$I_{2j(p,q)}^{(\alpha\beta)} = \frac{1}{A_{\alpha\beta}} (d_\alpha/2)^p \int_{-h_\beta/2}^{h_\beta/2} (\bar{x}_3^{(\beta)})^q [\sigma_{2j}^{(\alpha\beta)}(d_\alpha/2) + (-1)^{p+1}\sigma_{2j}^{(\alpha\beta)}(-d_\alpha/2)] \, d\bar{x}_3^{(\beta)}, \tag{18}$$

$$J_{3j(p,q)}^{(\alpha\beta)} = \frac{1}{A_{\alpha\beta}} (h_\beta/2)^q \int_{-d_\alpha/2}^{d_\alpha/2} (\bar{x}_2^{(\alpha)})^p [\sigma_{3j}^{(\alpha\beta)}(h_\beta/2) + (-1)^{q+1}\sigma_{3j}^{(\alpha\beta)}(-h_\beta/2)] \, d\bar{x}_2^{(\alpha)}, \tag{19}$$

$$A_{\alpha\beta} = d_\alpha h_\beta,$$

and $\delta_s^{(p)}$ represent the coefficients in the expansion of z^p in terms of the Legendre polynomials, i.e.

$$z^p = \delta_s^{(p)} P_s(z) \qquad s = 0, 1, \ldots, p \tag{20}$$

with $\delta_s^{(p)} = 0$ for $s > p$. Several values of $\delta_s^{(p)}$ can be found in ([8], p. 798).

In (18)–(19) $\sigma_{2j}^{(\alpha\beta)}(\pm d_\alpha/2)$ and $\sigma_{3j}^{(\alpha\beta)}(\pm h_\beta/2)$ stand for the interfacial stresses

$$\sigma_{2j}^{(\alpha\beta)}(x_i, \bar{x}_2^{(\alpha)} = \pm d_\alpha/2, \bar{x}_3^{(\beta)}, t) \text{ and } \sigma_{3j}^{(\alpha\beta)}(x_i, \bar{x}_2^{(\alpha)}, \bar{x}_3^{(\beta)} = \pm h_\beta/2, t)$$

respectively.

If the higher order stresses $S_{ij(p,q)}^{(\alpha\beta)}$ in (16) are expressed in terms of the higher order displacement gradients using the constitutive relations of the material, the dynamic eqns (16) for the Nth order EST form a system of $12(N+1)^2$ equations in $12(N+1)^2$ unknown $U_{j(m,n)}^{(\alpha\beta)}$ and 48 unknown interfacial stresses. By incorporating the 48 continuity conditions (11)–(14) for the displacements and stresses, it should be possible for a given order N to solve these equations subjected to the specific initial and boundary conditions.

BOUNDARY CONITIONS

The governing equations of motion of the homogeneous medium can be solved for a dynamic problem provided the relevant boundary conditions are formulated in the framework of the developed higher order continuum theory. This formulation which is given in the present section is similar to the procedure used in [9] for generating the boundary conditions for a second order EST for a laminated composite. It is based on Taylor expansions of the stresses and the applied loading functions about the center of every subcell of the representative cell.

To this end suppose that the tractions are prescribed on a plane whose normal is M_i. We introduce in each subcell a local coordinate system $(\bar{x}_1, \bar{x}_2^{(\alpha)}, \bar{x}_3^{(\beta)})$ aligned parallel to the global system (x_1, x_2, x_3), with its origin located on the exterior plane at the center of the subcell. With the same origin we also define a local orthogonal system $(\bar{n}, \bar{s}, \bar{b})$ with $\bar{n}$ in the direction normal to the plane, see Fig. 3.

The stress boundary conditions are given by

$$\hat{\sigma}_{ij}^{(\alpha\beta)}(\bar{x}_1, \bar{x}_2^{(\alpha)}, \bar{x}_3^{(\beta)}, t)|_{\bar{n}=0}M_j = \hat{\sigma}_{ij}^{(\alpha\beta)}(0, \bar{s}, \bar{b}, t)M_j = T_i^{(\alpha\beta)}(\bar{s}, \bar{b}, t) \qquad i, j = 1, 2, 3 \tag{21}$$

where $T_i^{(\alpha\beta)}(\bar{s}, \bar{b}, t)$ describe the distribution of the time-dependent traction components acting on the exterior plane.

Expanding $\hat{\sigma}_{ij}^{(\alpha\beta)}$ and $T_i^{(\alpha\beta)}$ in terms of $\bar{s}$, $\bar{b}$, up to the Nth power gives an equation which involve $\hat{\sigma}_{ij}^{(\alpha\beta)}$ and $T_i^{(\alpha\beta)}$ and their derivatives with respect to $\bar{s}$ and $\bar{b}$ all evaluated at the origin $\bar{n} = \bar{s} = \bar{b} = 0$. Equating like powers of $\bar{s}, \bar{b}$ yields a set of boundary conditions at that point.

If, on the other hand, the expansion (1) is substituted in the stress–strain relations, it is possible to express $\hat{\sigma}_{ij}^{(\alpha\beta)}$ and its derivatives evaluated at the origin $\bar{x}_1 = \bar{x}_2^{(\alpha)} = \bar{x}_3^{(\beta)} = 0$ in terms of the higher order displacement gradients. These expressions can be used in the above set of boundary conditions provided relations of the form

$$\frac{\partial}{\partial \bar{s}} = \frac{\partial \bar{x}_1}{\partial \bar{s}}\frac{\partial}{\partial \bar{x}_1} + \frac{\partial \bar{x}_2^{(\alpha)}}{\partial \bar{s}}\frac{\partial}{\partial \bar{x}_2^{(\alpha)}} + \frac{\partial \bar{x}_3^{(\beta)}}{\partial \bar{s}}\frac{\partial}{\partial \bar{x}_3^{(\beta)}} \tag{22}$$

are employed.

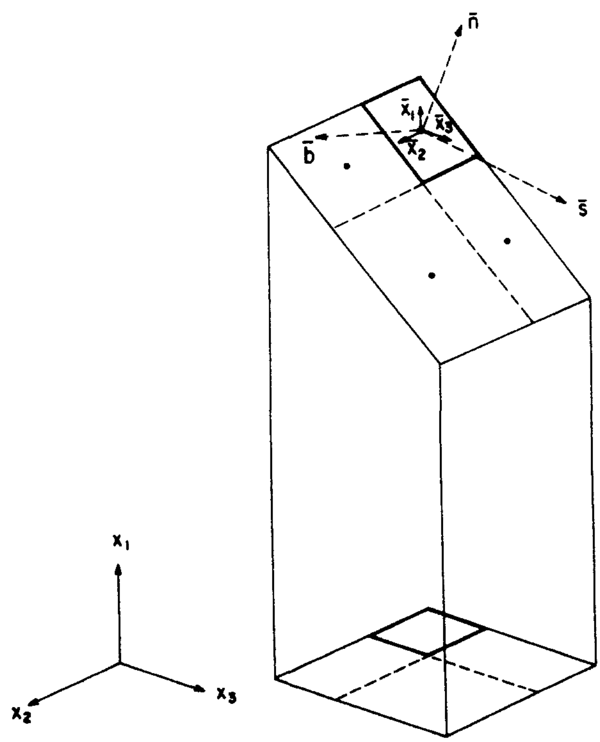

Fig. 3. Boundary coordinate systems for the exterior plane.

In conclusion, stress boundary conditions prescribed on the surface of a plane can be incorporated in the derived theory in the form of a set of relationships between the higher order displacement gradients and the time-dependent functions which describe the applied loadings. This formulation involves, however, the same restrictions mentioned in [9].

APPLICATION

We choose to apply the derived EST for fiber reinforced composites to obtain the dispersion curves for longitudinal time-harmonic waves propagating in the direction of the fibers and to compute the dynamic response of a fiber reinforced slab which is impacted on one of its surfaces uniformly in a direction parallel to the fibers. In both situations the composite material is modeled by a second order EST ($N = 2$). Accordingly, from symmetry considerations it can be seen that in both cases eqn (1) can be written in the form

$$\left.\begin{aligned} u_1^{(\alpha\beta)} &= W_{1(0,0)}^{(\alpha\beta)} + \tfrac{1}{2}(3\bar{x}_2^{(\alpha)2} - d_\alpha^2/4)W_{1(2,0)}^{(\alpha\beta)} + \tfrac{1}{2}(3\bar{x}_3^{(\beta)2} - h_\beta^2/4)W_{1(0,2)}^{(\alpha\beta)} \\ u_2^{(\alpha\beta)} &= \bar{x}_2^{(\alpha)}W_{2(1,0)}^{(\alpha\beta)} \\ u_3^{(\alpha\beta)} &= \bar{x}_3^{(\beta)}W_{3(0,1)}^{(\alpha\beta)} \end{aligned}\right\} \tag{23}$$

where

$$\begin{aligned} W_{1(0,0)}^{(\alpha\beta)} &= U_{1(0,0)}^{(\alpha\beta)} \\ W_{2(1,0)}^{(\alpha\beta)} &= U_{2(1,0)}^{(\alpha\beta)}/(d_\alpha/2) \\ W_{3(0,1)}^{(\alpha\beta)} &= U_{3(0,1)}^{(\alpha\beta)}/(h_\beta/2) \\ W_{1(2,0)}^{(\alpha\beta)} &= U_{1(2,0)}^{(\alpha\beta)}/(d_\alpha/2)^2 \\ W_{1(0,2)}^{(\alpha\beta)} &= U_{1(0,2)}^{(\alpha\beta)}/(h_\beta/2)^2. \end{aligned} \tag{24}$$

The above expressions for the displacements reflect the necessary properties that $u_1^{(\alpha\beta)}$ is even in $\bar{x}_2^{(\alpha)}$ and $\bar{x}_3^{(\beta)}$; $u_2^{(\alpha\beta)}$ is even in $\bar{x}_3^{(\beta)}$ and odd in $\bar{x}_2^{(\alpha)}$; and that $u_3^{(\alpha\beta)}$ is even in $\bar{x}_2^{(\alpha)}$ and odd in $\bar{x}_3^{(\beta)}$.

The strain components are computed from (23) according to (2) giving

$$\begin{aligned} \epsilon_{11}^{(\alpha\beta)} &= \partial_1 W_{1(0,0)}^{(\alpha\beta)} + \tfrac{1}{2}(3\bar{x}_2^{(\alpha)2} - d_\alpha^2/4)\partial_1 W_{1(2,0)}^{(\alpha\beta)} + \tfrac{1}{2}(3\bar{x}_3^{(\beta)2} - h_\beta^2/4)\partial_1 W_{1(0,2)}^{(\alpha\beta)}, \\ \epsilon_{22}^{(\alpha\beta)} &= W_{2(1,0)}^{(\alpha\beta)}, \ \epsilon_{33}^{(\alpha\beta)} = W_{3(0,1)}^{(\alpha\beta)}, \\ \epsilon_{12}^{(\alpha\beta)} &= \bar{x}_2^{(\alpha)}[3W_{1(2,0)}^{(\alpha\beta)} + \partial_1 W_{2(1,0)}^{(\alpha\beta)}]/2, \\ \epsilon_{13}^{(\alpha\beta)} &= \bar{x}_3^{(\beta)}[3W_{1(0,2)}^{(\alpha\beta)} + \partial_1 W_{3(0,1)}^{(\alpha\beta)}]/2, \\ \epsilon_{23}^{(\alpha\beta)} &= 0. \end{aligned} \tag{25}$$

Using the Hooke's law for perfectly elastic materials, the stress components in the subcell $(\alpha\beta)$ can be determined from

$$\sigma_{ij}^{(\alpha\beta)} = \lambda_{\alpha\beta}\epsilon_{kk}^{(\alpha\beta)}\delta_{ij} + 2\mu_{\alpha\beta}\epsilon_{ij}^{(\alpha\beta)} \tag{26}$$

where δ_{ij} is the Kronecker delta, $\lambda_{\alpha\beta}$ and $\mu_{\alpha\beta}$ are the Lame' constants of the material in the subcell. The stress components can be readily expressed in terms of the higher order displacement and displacement gradients using (25). These expressions can be used to evaluate the higher order stresses as follows:

$$S_{11(0,0)}^{(\alpha\beta)} = E_{\alpha\beta}\partial_1 W_{1(0,0)}^{(\alpha\beta)} + \lambda_{\alpha\beta}[W_{2(1,0)}^{(\alpha\beta)} + W_{3(0,1)}^{(\alpha\beta)}],$$

$$S_{11(2,0)}^{(\alpha\beta)} = \frac{d_\alpha^2}{12}\left[S_{11(0,0)}^{(\alpha\beta)} + \frac{d_\alpha^2}{10}\partial_1 W_{1(2,0)}^{(\alpha\beta)}\right],$$

$$S_{11(0,2)}^{(\alpha\beta)} = \frac{h_\beta^2}{12}\left[S_{11(0,0)}^{(\alpha\beta)} + \frac{h_\beta^2}{10}\partial_1 W_{1(0,2)}^{(\alpha\beta)}\right],$$

$$S_{12(1,0)}^{(\alpha\beta)} = \mu_{\alpha\beta}\frac{d_\alpha^2}{4}[W_{1(2,0)}^{(\alpha\beta)} + \partial_1 W_{2(1,0)}^{(\alpha\beta)}/3],$$

$$S_{13(0,1)}^{(\alpha\beta)} = \mu_{\alpha\beta}\frac{h_\beta^2}{4}[W_{1(0,2)}^{(\alpha\beta)} + \partial_1 W_{3(0,1)}^{(\alpha\beta)}/3],$$

$$S_{22(0,0)}^{(\alpha\beta)} = \lambda_{\alpha\beta}[\partial_1 W_{1(0,0)}^{(\alpha\beta)} + W_{3(0,1)}^{(\alpha\beta)}] + E_{\alpha\beta} W_{2(1,0)}^{(\alpha\beta)},$$

$$S_{33(0,0)}^{(\alpha\beta)} = \lambda_{\alpha\beta}[\partial_1 W_{1(0,0)}^{(\alpha\beta)} + W_{2(1,0)}^{(\alpha\beta)}] + E_{\alpha\beta} W_{3(0,1)}^{(\alpha\beta)} \tag{27}$$

where $E_{\alpha\beta} = \lambda_{\alpha\beta} + 2\mu_{\alpha\beta}$.

The displacement continuity conditions (11)–(12) yield in the present situation the 8 relations

$$\left.\begin{aligned} W_{1(0,0)}^{(1\beta)} + (d_1^2/4) W_{1(2,0)}^{(1\beta)} &= W_{1(0,0)}^{(2\beta)} + (d_2^2/4) W_{1(2,0)}^{(2\beta)}, \\ d_1 W_{2(1,0)}^{(1\beta)} &= -d_2 W_{2(1,0)}^{(2\beta)}, \\ W_{1(0,0)}^{(\alpha 1)} + (h_1^2/4) W_{1(0,2)}^{(\alpha 1)} &= W_{1(0,0)}^{(\alpha 2)} + (h_2^2/4) W_{1(0,2)}^{(\alpha 2)}, \\ h_1 W_{3(0,1)}^{(\alpha 1)} &= -h_2 W_{3(0,1)}^{(\alpha 2)}. \end{aligned}\right\} \tag{28}$$

The dynamic equations of motion are obtained from (16) yielding in the present case 20 equations of the form

$$\partial_1 S_{11(0,0)}^{(\alpha\beta)} + I_{21(0,0)}^{(\alpha\beta)} + J_{31(0,0)}^{(\alpha\beta)} = \rho_{\alpha\beta}\ddot{W}_{1(0,0)}^{(\alpha\beta)} \tag{29}$$

$$\partial_1 S_{12(1,0)}^{(\alpha\beta)} + I_{22(1,0)}^{(\alpha\beta)} + J_{32(1,0)}^{(\alpha\beta)} - S_{22(0,0)}^{(\alpha\beta)} = \rho_{\alpha\beta}\frac{d_\alpha^2}{12}\ddot{W}_{2(1,0)}^{(\alpha\beta)} \tag{30}$$

$$\partial_1 S_{13(0,1)}^{(\alpha\beta)} + I_{23(0,1)}^{(\alpha\beta)} + J_{33(0,1)}^{(\alpha\beta)} - S_{33(0,0)}^{(\alpha\beta)} = \rho_{\alpha\beta}\frac{h_\beta^2}{12}\ddot{W}_{3(0,1)}^{(\alpha\beta)} \tag{31}$$

$$\partial_1 S_{11(2,0)}^{(\alpha\beta)} + \frac{d_\alpha^2}{4} I_{21(0,0)}^{(\alpha\beta)} + J_{31(2,0)}^{(\alpha\beta)} - 2S_{21(1,0)}^{(\alpha\beta)} = \rho_{\alpha\beta}\frac{d_\alpha^2}{12}\left[\ddot{W}_{1(0,0)}^{(\alpha\beta)} + \frac{d_\alpha^2}{10}\ddot{W}_{1(2,0)}^{(\alpha\beta)}\right], \tag{32}$$

$$\partial_1 S_{11(0,2)}^{(\alpha\beta)} + I_{21(0,2)}^{(\alpha\beta)} + \frac{h_\beta^2}{4} J_{31(0,0)}^{(\alpha\beta)} - 2S_{31(0,1)}^{(\alpha\beta)} = \rho_{\alpha\beta}\frac{h_\beta^2}{12}\left[\ddot{W}_{1(0,0)}^{(\alpha\beta)} + \frac{h_\beta^2}{10}\ddot{W}_{1(0,2)}^{(\alpha\beta)}\right]. \tag{33}$$

Due to the existing symmetry in the present problem, $\sigma_{31}^{(\alpha\beta)}$ should be even in $\bar{x}_2^{(\alpha)}$ and odd in $\bar{x}_3^{(\beta)}$ so that to the order of expansion used in the second order theory $\sigma_{31}^{(\alpha\beta)}$ must be independent of $\bar{x}_2^{(\alpha)}$. Accordingly $J_{31(2,0)}^{(\alpha\beta)} = d_\alpha^2 J_{31(0,0)}^{(\alpha\beta)}/12$. Similarly, $\sigma_{21}^{(\alpha\beta)}$ is independent of $\bar{x}_3^{(\beta)}$ so that $I_{21(0,2)}^{(\alpha\beta)} = h_\beta^2 I_{21(0,0)}^{(\alpha\beta)}/12$. Finally, $\sigma_{23}^{(\alpha\beta)} = \sigma_{32}^{(\alpha\beta)}$ must be odd in both $\bar{x}_2^{(\alpha)}$ and $\bar{x}_3^{(\beta)}$ so that $I_{23(0,1)}^{(\alpha\beta)} = J_{32(1,0)}^{(\alpha\beta)} = 0$.

From the continuity conditions of the stresses (13)–(14) we have

$$
\begin{aligned}
I^{(12)}_{21(0,0)} &= -I^{(22)}_{21(0,0)}, \qquad & J^{(12)}_{31(0,0)} &= -J^{(11)}_{31(0,0)},\\
I^{(21)}_{21(0,0)} &= -I^{(11)}_{21(0,0)}, \qquad & J^{(21)}_{31(0,0)} &= -J^{(22)}_{31(0,0)},\\
I^{(12)}_{22(1,0)} &= I^{(22)}_{22(1,0)}, \qquad & J^{(12)}_{33(0,1)} &= J^{(11)}_{33(0,1)},\\
I^{(21)}_{22(1,0)} &= I^{(11)}_{22(1,0)}, \qquad & J^{(21)}_{33(0,1)} &= J^{(22)}_{33(0,1)}.
\end{aligned}
\tag{34}
$$

Consequently in the system of 20 equations given in (29)–(33) there are 20 unknown higher order displacements and 8 unknown interfacial stresses $I^{(11)}_{21(0,0)}$, $I^{(22)}_{21(0,0)}$, $J^{(11)}_{31(0,0)}$, $J^{(22)}_{31(0,0)}$, $I^{(11)}_{22(1,0)}$, $I^{(22)}_{22(1,0)}$, $J^{(11)}_{33(0,1)}$, and $J^{(22)}_{33(0,1)}$. By incorporating the 8 displacement continuity relations given by (28) we can solve the above system subject to initial and boundary conditions.

For the problem of an impacted slab occupying the region $0 \le x_1 \le H$; $-\infty < x_2, x_3 < \infty$ and subjected to an extended uniform time-dependent tractions on the plane $x_1 = 0$ (which is perpendicular to the fibers) while the other surface is kept rigidly clamped, the boundary conditions (21) simplify to

$$\hat{\sigma}^{(\alpha\beta)}_{1j}(0, \bar{x}^{(\alpha)}_2, \bar{x}^{(\beta)}_3, t) = T_j(t) \qquad j = 1, 2, 3 \tag{35}$$

with $T_2(t) = T_3(t) = 0$.

Expanding the l.h.s. of (35) in power series about the center of the subcell up to the second order and equating equal powers gives

$$
\begin{aligned}
&\hat{\sigma}^{(\alpha\beta)}_{1j}\Big|_0 = T_j(t),\\
&\frac{\partial}{\partial \bar{x}^{(\alpha)}_2}\hat{\sigma}^{(\alpha\beta)}_{1j}\Big|_0 = 0, \qquad \frac{\partial^2}{\partial \bar{x}^{(\beta)2}_3}\hat{\sigma}^{(\alpha\beta)}_{1j}\Big|_0 = 0.\\
&\frac{\partial^2}{\partial \bar{x}^{(\alpha)2}_2}\hat{\sigma}^{(\alpha\beta)}_{1j}\Big|_0 = 0, \qquad \frac{\partial^2}{\partial \bar{x}^{(\beta)2}_3}\hat{\sigma}^{(\alpha\beta)}_{1j}\Big|_0 = 0.
\end{aligned}
\tag{36}
$$

From (36) we obtain the following non-trivial conditions imposed at $x_1 = 0$

$$E_{\alpha\beta}\left[\partial_1 W^{(\alpha\beta)}_{1(0,0)} - \frac{d^2_\alpha}{8}\partial_1 W^{(\alpha\beta)}_{1(2,0)} - \frac{h^2_\beta}{8}\partial_1 W^{(\alpha\beta)}_{1(0,2)}\right] + \lambda_{\alpha\beta}[W^{(\alpha\beta)}_{2(1,0)} + W^{(\alpha\beta)}_{3(0,1)}] = T_1(t),$$

$$\partial_1 W^{(\alpha\beta)}_{1(2,0)} = 0, \qquad \partial_1 W^{(\alpha\beta)}_{1(0,2)} = 0, \qquad 3W^{(\alpha\beta)}_{1(2,0)} + \partial_1 W^{(\alpha\beta)}_{2(1,0)} = 0, \qquad 3W^{(\alpha\beta)}_{1(0,2)} + \partial_1 W^{(\alpha\beta)}_{3(0,1)} = 0 \tag{37}$$

Eqns (37) form a system of 20 boundary conditions for the 20 higher order displacements at $x_1 = 0$.

On the rigidly clamped surface $x_1 = H$ we have

$$W^{(\alpha\beta)}_{1(0,0)} = W^{(\alpha\beta)}_{2(1,0)} = W^{(\alpha\beta)}_{3(0,1)} = W^{(\alpha\beta)}_{1(2,0)} = W^{(\alpha\beta)}_{1(0,2)} = 0. \tag{38}$$

RESULTS

Results are given for a fiber reinforced composite of circular silica fibers (1) and polystyrene matrix (2) whose mechanical and geometrical parameters are given in Table 1. This table is taken from Ref. [7] where the finite element method was employed to compute the dispersion curves of the composite material and compared with some experimental data. The length ratio is computed from the relation $d_1^2 = h_1^2 = \pi a^2$ so that $d_1/d = h_1/d = a\pi^{1/2}/d$.

(1) *Dispersion curve for the lowest longitudinal mode*

In order to investigate the dispersion of longitudinal harmonic waves propagating in an infinite medium in the direction of the fibers, we substitute for the 28 dependent variables in

Table 1. Mechanical and geometric properties of the fiber reinforced composite: silica(1)–polystyrene (2)

$\lambda_1 = 1.607 \times 10^{11}$ dyne/cm^2	$\lambda_2 = 0.317 \times 10^{11}$ dyne/cm^2
$\mu_1 = 3.12 \times 10^{11}$ dyne/cm^2	$\mu_2 = 0.1323 \times 10^{11}$ dyne/cm^2
$\rho_1 = 2.2$ gm/cm^3	$\rho_2 = 1.056$ gm/cm^3

fiber radius = a = 0.051 cm
fiber spacing = $d = d_1 + d_2 = h_1 + h_2 = 0.236$ cm.

(29)–(33) travelling waves of the form $A_r \exp[ik(x_1 - ct)]$ where A_r ($r = 1, \ldots, 28$) are amplitude factors, k is the wave number, c is the phase velocity and i is the imaginary unit. This gives a system of 28 homogeneous equations for A_r so that for a non-trivial solution the determinant of the matrix of coefficients must vanish, yielding the frequency equation

$$\det[\mathbf{A}] = 0. \tag{39}$$

The elements of the matrix A are given in the Appendix.

In Fig. 3 the lowest (acoustical) dispersion curve computed from (39) is shown and compared with the finite element and experimental results presented in [7]. For this comparison the dimensionless phase velocity $c/(c_T)_1$, where $(c_T)_1 = (\mu_1/\rho_1)^{1/2}$, is plotted vs kd. It is well seen that excellent agreement is obtained over the shown range of frequencies when using a second order EST. For the smaller wave lengths it is possible to increase the accuracy further by adopting a higher order EST.

For vanishingly small wave number the phase velocity has the value $c = 0.96(c_T)_1$. For this limiting case this value can be obtained from the effective modulus theory in which the fiber reinforced composite is modeled by a homogeneous but transversely isotropic elastic material. Here the stress–strain relation for a longitudinal one-dimension motion (uniaxial strain) in the direction of the fibers (i.e. the direction of the axis of symmetry) can be written in the form

$$\sigma_{11} = c^{\dagger}_{11}\epsilon_{11} \tag{40}$$

where

$$c^{\dagger}_{11} = E^*_A + 4k^*(\nu^*_A)^2. \tag{41}$$

In (41) the effective moduli E^*_A, k^*, ν^*_A can be expressed in terms of the elastic constants of the matrix and fibers and their volume concentrations. They are given by ([10], p. 131).

$$k^* = k_1 + V_2/[1/(k_2 - k_1) + V_1/(k_1 + \mu_1)],$$

$$E^*_A = Y_1V_1 + Y_2V_2 + 4(\nu_2 - \nu_1)^2V_1V_2/[V_1/k_2 + V_2/k_1 + 1/\mu_1],$$

$$\nu^*_A = \nu_1V_1 + \nu_2V_2 + (\nu_2 - \nu_1)(1/k_1 - 1/k_2)V_1V_2/[V_1/k_2 + V_2/k_1 + 1/\mu_1] \tag{42}$$

where Y_i, ν_i are the Young's moduli and Poisson's ratios of the materials, V_i are their volume fractions and $k_i = \mu_i/(1 - 2\nu_i)$, $i = 1, 2$. Accordingly, the effective longitudinal wave speed can be computed from $c^* = [c^{\dagger}_{11}/\rho^*]^{1/2}$ with $\rho^* = V_1\rho_1 + V_2\rho_2$ being the effective density. For the composite material described in Table 1 we obtain $c^* = 0.96(c_T)_1$ which is the same value obtained before. Consequently, eqn (39) can be regarded as an implicit expression for the effective modulus in terms of the mechanical and geometrical properties of the constituents, since its lowest root for a vanishingly small wave number gives the value of this modulus. Obviously, the other effective moduli can be determined essentially by the same procedure. It should be noticed, however, that it might be possible to compute the effective moduli of a fiber reinforced composite by employing a lower order EST than the second used here.

(2) *Transient wave propagation*

For the problem of an impacted slab which is initially at rest, the equations of motion (29)–(33), the continuity conditions (28) and (34) together with the boundary conditions (37)–(38)

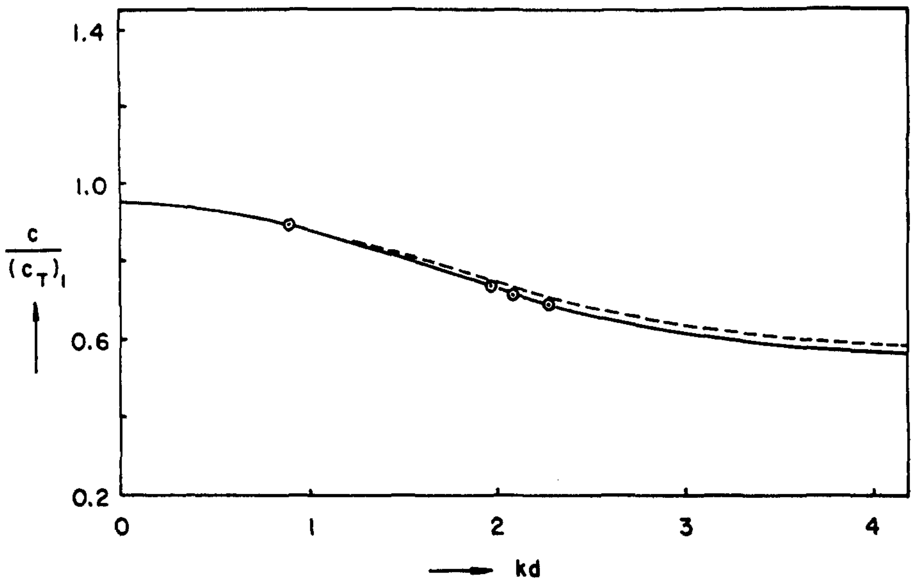

Fig. 4. Theoretical (-----), numerical (———) and experimental (·····) phase velocities for longitudinal waves propagating in the direction of the fibers.

determine completely the motion. The applied input on the surface $x_1 = 0$ is described by $T_1(t)$ in (37) which is chosen here in the form

$$T_1(t) = \begin{cases} \mu_1 \quad \sin \pi t/(2t_m) & t \le t_m \\ \mu_1 & t > t_m \end{cases} \tag{43}$$

where t_m is an appropriate constant.

The one-dimensional equations of motion were solved by a finite difference procedure and

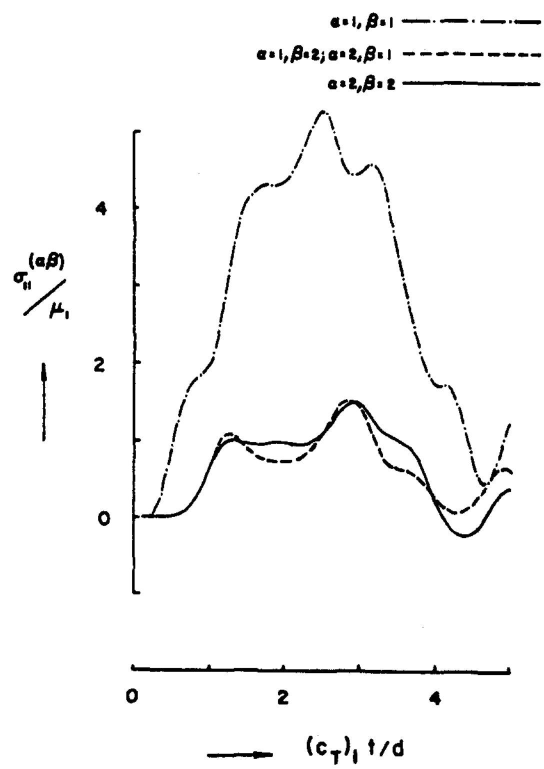

Fig. 5. For the loading function (43) with $(c_T)_1 t_m/d = 0.5$ and slab width $H = d$, the stresses $\sigma_{11}^{(\alpha\beta)}$ are shown at $x_1 = H/2$, $\bar{x}_2^{(\alpha)} = \bar{x}_2^{(\beta)} = 0$ versus time.

results for the time dependent stresses $\sigma_{11}^{(\alpha\beta)}(x_1, \bar{x}_2^{(\alpha)} = 0, \bar{x}_3^{(\beta)} = 0, t)$ are shown in Fig. 4 at the mid-point $x_1 = H/2$ vs the non-dimensional time $(c_T)_1 t/d$. The computations were performed with $(c_T)_1 t_m/d = 0.5$ and $H = d$. It should be noticed that due to the existing symmetry in the geometry of the problem $\sigma_{11}^{(12)} = \sigma_{11}^{(21)}$. The curves in the figure exhibit both the direct and reflected waves and it can be noticed that the propagated stress levels can exceed by several times the stress applied on the boundary especially in the stiff material.

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Appendix

The nonzero elements of the matrix $\underset{\sim}{A}$ in (39) are given by:

$A(1,1) = \rho_1 \xi + E_1 \eta$, $A(1,2) = A(1,3) = -\lambda_1 k$,

$A(1,21) = c_1$, $A(1,22) = c_2$,

$A(2,1) = -\lambda_1 k$, $A(2,2) = c_3 \xi + c_4 \eta - E_1$, $A(2,3) = -\lambda_1$,

$A(2,4) = c_5 k$, $A(2,25) = 0.5$,

$A(3,1) = -\lambda_1 k$, $A(3,2) = -\lambda_1$, $A(3,3) = c_6 \xi + c_7 \eta - E_1$,

$A(3,5) = c_8 k$, $A(3,26) = 0.5$,

$A(4,1) = c_3 \xi + c_9 \eta$, $A(4,2) = (c_{11} - c_{10})k$, $A(4,3) = -c_{10} k$,

$A(4,4) = c_{12} \xi + c_{13} \eta - c_{14}$, $A(4,21) = c_{15}$, $A(4,22) = c_{16}$,

$A(5,1) = c_6 \xi + c_{17} \eta$, $A(5,2) = -c_{18} k$,

$A(5,3) = (c_{19} - c_{18})k$, $A(5,5) = c_{20} \xi + c_{21} \eta - c_{22}$,

$A(5,21) = c_{23}$, $A(5,22) = c_{24}$,

$A(6,6) = \rho_2 \xi + E_2 \eta$, $A(6,7) = A(6,8) = -\lambda_2 k$,

$A(6,23) = -c_1$, $A(6,22) = -c_{25}$,

$A(7,6) = -\lambda_2 k$, $A(7,7) = c_{26} \xi + c_{27} \eta - E_2$, $A(7,8) = -\lambda_2$,

$A(7,9) = c_{28} k$, $A(7,27) = 0.5$,

$A(8,6) = -\lambda_2 k$, $A(8,7) = -\lambda_2$, $A(8,8) = c_{29} \xi + c_{30} \eta - E_2$,

$A(8,10) = c_{31} k$, $A(8,26) = 0.5$,

$A(9,6) = c_{26} \xi + c_{32} \eta$, $A(9,7) = (c_{33} - c_{34})k$, $A(9,8) = -c_{34} k$,

$A(9,9) = c_{68} \xi + c_{69} \xi - c_{70}$, $A(9,22) = -c_{35}$, $A(9,23) = -c_{15}$,

$A(10,6) = c_{29}\,\xi + c_{36}\,\eta$, $A(10,7) = -\,c_{37}\,k$,

$A(10,8) = (c_{38} - c_{37})k$, $A(10,10) = c_{39}\,\xi + c_{40}\,\eta - c_{41}$,

$A(10,22) = -\,c_{52}$, $A(10,23) = -\,c_{42}$,

$A(11,11) = A(6,6)$, $A(11,12) = A(6,7)$, $A(11,13) = A(11,12)$,

$A(11,21) = -\,c_{43}$, $A(11,24) = -\,c_{2}$,

$A(12,11) = A(11,12)$, $A(12,12) = c_{44}\,\xi + c_{45}\,\eta - E_2$,

$A(12,13) = -\,\lambda_2$, $A(12,14) = c_{46}\,k$, $A(12,25) = 0.5$,

$A(13,11) = A(11,12)$, $A(13,12) = -\,\lambda_2$,

$A(13,13) = c_{47}\,\xi + c_{48}\,\eta - E_2$, $A(13,15) = c_{49}\,k$, $A(13,25) = 0.5$,

$A(14,11) = c_{44}\,\xi + c_{50}\,\eta$, $A(14,12) = (c_{51} - c_{52})k$,

$A(14,13) = -\,c_{52}\,k$, $A(14,14) = c_{53}\,\xi + c_{54}\,\eta - c_{55}$,

$A(14,21) = -c_{56}$, $A(14,24) = -\,c_{57}$,

$A(15,11) = c_{47}\,\xi + c_{58}\,\eta$, $A(15,12) = -\,c_{59}\,k$,

$A(15,13) = (c_{60} - c_{59})\,k$, $A(15,15) = c_{61}\,\xi + c_{62}\,\eta - c_{63}$,

$A(15\ 21) = -c_{64}$, $A(15,24) = -\,c_{24}$,

$A(16,16) = A(6,6)$, $A(16,17) = A(16,18) = A(6,7)$,

$A(16,23) = c_{43}$, $A(16,24) = c_{25}$,

$A(17,16) = A(16,17)$, $A(17,17) = A(12,12)$, $A(17,18) = -\,\lambda_2$,

$A(17,19) = A(12,14)$, $A(17,27) = 0.5$,

$A(18,16) = A(16,17)$, $A(18,17) = -\,\lambda_2$, $A(18,18) = A(8,8)$,

$A(18,20) = A(8,10)$, $A(18,28) = 0.5$,

$A(19,16) = A(14,11)$, $A(19,17) = A(14,12)$, $A(19,18) = A(14,13)$,

$A(19,19) = A(14,14)$, $A(19,23) = c_{56}$, $A(19,24) = c_{65}$,

$A(20,16) = A(10,6)$, $A(20,17) = A(10,7)$, $A(20,18) = A(10,8)$,

$A(20,20) = A(10,10)$, $A(20,23) = c_{66}$, $A(20,24) = c_{67}$,

$A(21,1) = 1$, $A(21,4) = d_1^2/4$, $A(21,11) = -\,1$, $A(21,14) = -\,d_2^2/4$,

$A(22,6) = 1$, $A(22,9) = d_1^2/4$, $A(22,16) = -\,1$, $A(22,19) = -\,d_2^2/4$,

$A(23,1) = 1$, $A(23,5) = h_1^2/4$, $A(23,6) = -\,1$, $A(23,10) = -\,h_2^2/4$,

$A(24,11) = 1$, $A(24,15) = h_1^2/4$, $A(24,16) = -\,1$, $A(24,20) = -\,h_2^2/4$,

$A(25,2) = d_1$, $A(25,12) = d_2$,

$A(26,3) = h_1$, $A(26,8) = h_2$,

$A(27,7) = d_1$, $A(27,17) = d_2$,

$A(28,13) = h_1$, $A(28,18) = h_2$,

where $\xi = k^2c^2$ and $\eta = -\,k^2$.

In these equations:

$c_1 = 1/d_1$, $c_2 = 1/h_1$, $c_3 = \rho_1\,d_1^2/12$, $c_4 = \mu_1\,d_1^2/12$,

$c_5 = \mu_1\,d_1^2/4$, $c_6 = \rho_1\,h_1^2/12$, $c_7 = \mu_1\,h_1^2/12$, $c_8 = \mu_1\,h_1^2/4$,

$c_9 = E_1\,d_1^2/12$, $c_{10} = \lambda_1\,d_1^2/12$, $c_{11} = 2c_4$, $c_{12} = \rho_1\,d_1^4/120$,

$c_{13} = E_1\,d_1^4/120$, $c_{14} = 2c_5$, $c_{15} = d_1/4$, $c_{16} = d_1^2/(12\,h_1)$,

$c_{17} = E_1\, h_1^2/12$, $c_{18} = \lambda_1\, h_1^2/12$, $c_{19} = 2c_7$, $c_{20} = \rho_1\, h_1^4/120$,

$c_{21} = E_1\, h_1^4/120$, $c_{22} = 2c_8$, $c_{23} = h_1^2/(12\, d_1)$, $c_{24} = h_1/4$,

$c_{25} = 1/h_2$, $c_{26} = \rho_2\, d_1^2/12$, $c_{27} = \mu_2\, d_1^2/12$, $c_{28} = 3c_{27}$,

$c_{29} = \rho_2\, h_2^2/12$, $c_{30} = \mu_2\, h_2^2/12$, $c_{31} = 3c_{30}$, $c_{32} = E_2\, d_1^2/12$,

$c_{33} = 2c_{27}$, $c_{34} = \lambda_2\, d_1^2/12$, $c_{35} = d_1^2/(12\, h_2)$, $c_{36} = E_2\, h_2^2/12$,

$c_{37} = \lambda_2\, h_2^2/12$, $c_{38} = 2c_{30}$, $c_{39} = \rho_2\, h_2^4/120$, $c_{40} = E_2\, h_2^4/120$,

$c_{41} = 2c_{31}$, $c_{42} = h_2^2/(12\, d_1)$, $c_{43} = 1/d_2$, $c_{44} = \rho_2\, d_2^2/12$,

$c_{45} = \mu_2\, d_2^2/12$, $c_{46} = 3c_{45}$, $c_{47} = \rho_2\, h_1^2/12$, $c_{48} = \mu_2\, h_1^2/12$,

$c_{49} = 3c_{48}$, $c_{50} = E_2\, d_2^2/12$, $c_{51} = 2c_{45}$, $c_{52} = \lambda_2\, d_2^2/12$,

$c_{53} = \rho_2\, d_2^4/120$, $c_{54} = E_2\, d_2^4/120$, $c_{55} = 2c_{46}$, $c_{56} = d_2/4$,

$c_{57} = d_2^2/(12\, h_1)$, $c_{58} = E_2\, h_1^2/12$, $c_{59} = \lambda_2\, h_1^2/12$, $c_{60} = 2c_{48}$,

$c_{61} = \rho_2\, h_1^4/120$, $c_{62} = E_2\, h_1^4/120$, $c_{63} = 2c_{49}$, $c_{64} = h_1^2/(12\, d_2)$,

$c_{65} = d_2^2/(12\, h_2)$, $c_{66} = h_2^2/(12\, d_2)$, $c_{67} = h_2/4$, $c_{68} = \rho_2\, d_1^4/120$,

$c_{69} = E_2\, d_1^4/120$, $c_{70} = 2c_{28}$.